Filling Chessboards to Capacity with Mutually Non-attacking Pieces

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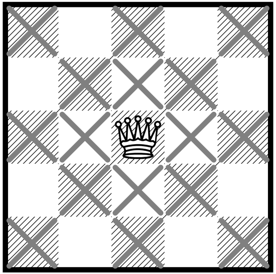
Outline

Section 1

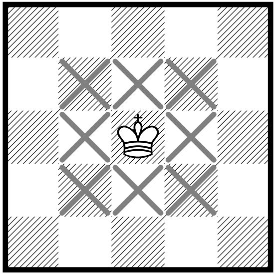
Introduction

Chess piece attacks



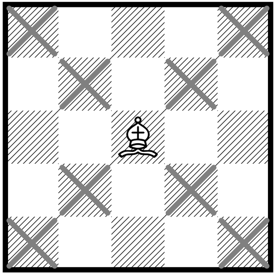
Queen attacks row, column, and diagonals.

Chess piece attacks



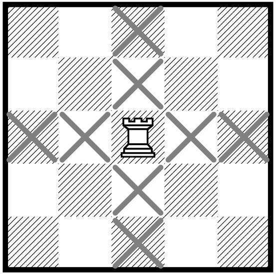
King attacks neighboring squares.

Chess piece attacks



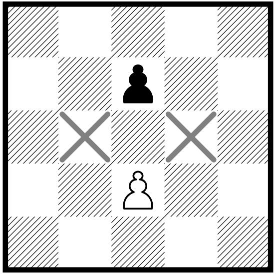
Bishop attacks diagonals.

Chess piece attacks



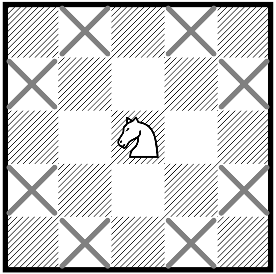
Rook attacks row and column.

Chess piece attacks



White pawns attack diagonally forward.
Black pawns attack diagonally backward.

Chess piece attacks



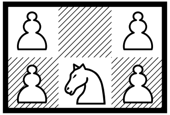
Knights attack the opposite corner of 2×3 or 3×2 array.

What is capacity?

Definition

Given a set S of allowed piece types, the **capacity** *of an $m \times n$ chessboard (w.r.t. S) is the maximum number of allowed pieces we can put on the board such that no piece attacks another.*

For example, if the allowed pieces are knights and white pawns, then the capacity of a 2×3 board is 5.



Section 2

Calculating Capacities for Small Boards

0-1 Integer programming formulation

Suppose the allowed pieces are white and black pawns.
For each row i and column j, let

$$w_{i,j} = \begin{cases} 1 & \text{if there is a white pawn on square } (i,j) \\ 0 & \text{otherwise} \end{cases}$$

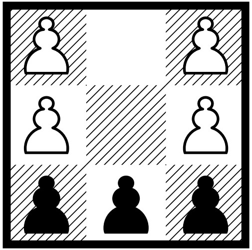
$$b_{i,j} = \begin{cases} 1 & \text{if there is a black pawn on square } (i,j) \\ 0 & \text{otherwise} \end{cases}$$

$$p_{i,j} = \begin{cases} 1 & \text{if there is a pawn on square } (i,j) \\ 0 & \text{otherwise} \end{cases}$$

0-1 IP continued

Maximize $\sum_{i,j} p_{i,j}$ subject to, for each possible i and j,

$$p_{i,j} = w_{i,j} + b_{i,j}$$
$$p_{i-1,j-1} + w_{i,j} \leq 1$$
$$p_{i-1,j+1} + w_{i,j} \leq 1$$
$$p_{i+1,j-1} + b_{i,j} \leq 1$$
$$p_{i+1,j+1} + b_{i,j} \leq 1$$
$$p_{i,j}, w_{i,j}, b_{i,j} \in \{0, 1\}$$



Some capacities (pawns only)

$m\backslash n$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	4	7	8	11	12	15	16	19
4	4	6	10	12	16	18	22	24	28
5	5	8	12	16	20	24	28	32	36
6	6	8	14	16	22	24	30	32	38
7	7	10	16	20	26	30	36	40	46
8	8	12	18	24	30	36	42	48	54
9	9	12	20	24	32	36	44	48	56

Table 1: Capacity (white and black pawns) of $m \times n$ boards

Some capacities (standard chess pieces)

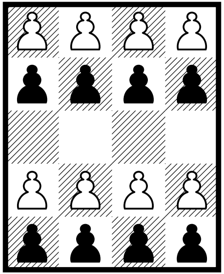
$m \backslash n$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	**6**	**8**	8	11	**13**	**16**	16	19
4	4	6	10	12	16	18	22	24	28
5	5	8	12	16	20	24	28	32	36
6	6	**10**	14	16	22	**25**	30	32	38
7	7	**12**	16	20	26	30	36	40	46
8	8	12	18	24	30	36	42	48	54
9	9	**14**	20	24	32	**37**	44	48	56

Table 2: Capacity (using all standard pieces) of $m \times n$ boards

Section 3

Patterns for Lower Bounds

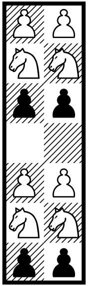
m-layer pattern



Proposition

The capacity of an $m \times n$ *board is at least* $T = T(m, n) = \lceil \frac{2m}{3} \rceil n$.

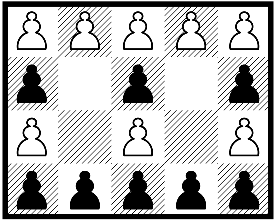
Knighted m-layer pattern



Proposition

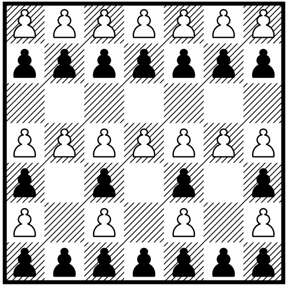
The capacity of an $m \times 2$ *board is at least* $2\lceil \frac{3m}{4} \rceil$.

Loopy pattern



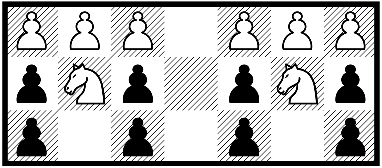
- On $4 \times n$ board with n odd, occupies $3n + 1 = T + 1$ squares.
- On $6 \times n$ board with n odd, occupies $4n + 2 = T + 2$ squares.

Loopy pattern covered by k roofs



- On $(3k+4) \times n$ board, where n is odd, occupies $(2k+3)n+1 = T+1$ squares.
- On $(3k+6) \times n$ board, where n is odd, occupies $(2k+4)n+2 = T+2$ squares.

Horse stable pattern



On $3 \times n$ board, occupies

- $2n$ squares if $n \equiv 0 \pmod 4$,
- $2n+1$ squares if $n \equiv 1$ or $2 \pmod 4$, and
- $2n+2$ squares if $n \equiv 3 \pmod 4$.

Section 4

Upper Bounds

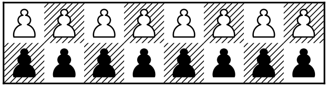
One row, one column, and two rows

Proposition

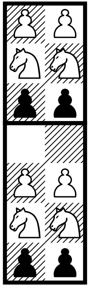
The capacity of a $1 \times n$ or an $n \times 1$ board is n.

Proposition

The capacity of a $2 \times n$ board is $2n$.



Two columns



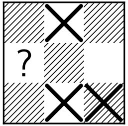
Proposition

The capacity of an $m \times 2$ *board is* $2\left\lceil \frac{3m}{4} \right\rceil$.

Three columns

Proposition

The capacity of a 3×3 board is 8.



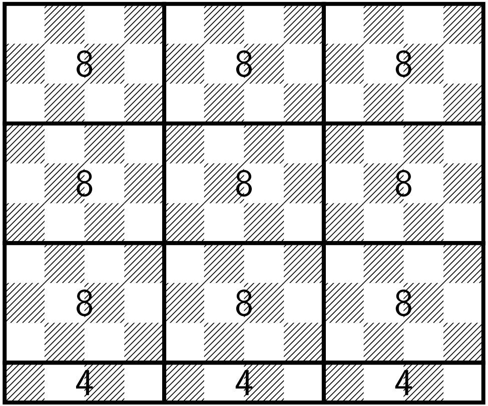
Proposition

For $m > 1$, the capacity of a $m \times 3$ board is $2m + 2$.

$4b$ columns

Proposition

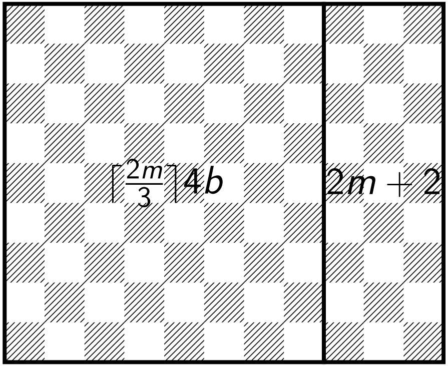
If n is a multiple of 4, then the capacity of an $m \times n$ board is $\lceil \frac{2m}{3} \rceil n$.



$4b + 3$ columns

Proposition

Let $m = 3a + i$ with $a \geq 2$ and $i = 0, 1,$ or 2 and let $n = 4b + 3$ with $b \geq 1$. The capacity of a $m \times n$ board is $\lceil \frac{2m}{3} \rceil n + (2 - i)$.



- $i = 0$: stable with $a - 1$ roofs
- $i = 1$: loopy with $a - 1$ roofs
- $i = 2$: m-layer

Section 5

Open Problems

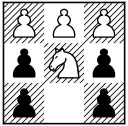
Conjectured capacities

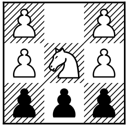
$m \backslash n$	$4b$	$4b+1$	$4b+2$	$4b+3$
3	$T = 2n$ Stable	T+1 Stable	T+1 Stable	T+2 Stable
$3a$ $a \geq 2$	$T = 2an$ Stable with $a-1$ roofs	T+2 Loopy with $a-2$ roofs	T+1 Stable with $a-1$ roofs	T+2 Stable with $a-1$ roofs
$3a+1$	$T = (2a+1)n$ m-layer	T+1 Loopy with $a-1$ roofs	T m-layer	T+1 Loopy with $a-1$ roofs
$3a+2$	$T = (2a+2)n$ m-layer	T m-layer	T m-layer	T m-layer

Table 3: Conjectured capacity of $m \times n$ board for $m \geq 3$, $n \geq 4$, where $T = \lceil \frac{2m}{3} \rceil n$

Number of solutions

- For a given $m \times n$ board with capacity c, how many arrangements of c pieces are there?
  - For the $1 \times p$ or $p \times 1$ board, there are 4^p arrangements.
  - For the $2 \times n$ board, there is 1 arrangement.
  - For the 3×3 board, there are 2 arrangements:





  - For the $(3a+2) \times 4b$ board, there is 1 arrangement: the $(3a+2)$-layer pattern.

Number of solutions, continued

$m \backslash n$	2	3	4	5	6	7	8
2	16	4	1	1	1	1	1
3	1	2	1826	203	44	4	252312
4	108	16	337	64	1728	256	9728
5	34	68	1	1	1	1	1
6	8	264	31779	4096	48	65544	31380137
7	1	1092	696	128	4704	512	28544
8	233	4420	1	1	1	1	1

Table 4: Numbers of ways to fill the capacity of $m \times n$ boards

Alternative piece sets

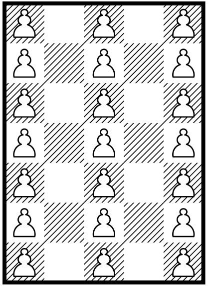
- Sets without pawns, like {Rook, Bishop, King}

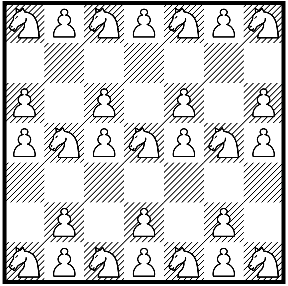
$m \backslash n$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	2	4	4	6	6	8	8
3	3	4	4	6	7	8	9	10
4	4	4	6	6	8	8	10	11
5	5	6	7	8	9	10	12	13
6	6	6	8	8	10	10	12	13
7	7	8	9	10	12	12	16	16
8	8	8	10	11	13	13	16	17

Table 5: Capacity (rooks, bishops, kings) of $m \times n$ boards

Alternative piece sets

- Standard set without black pawns



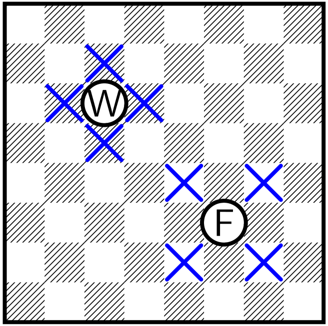


Conjecture

For $n \geq 3$, the capacity of an $m \times n$ board w.r.t. the standard chess pieces of one side is $\max(m\lceil \frac{n}{2} \rceil, \lceil \frac{n(m+1)}{2} \rceil)$.

Alternative piece sets

- Sets with nonstandard pieces, like the Wazir or Ferz

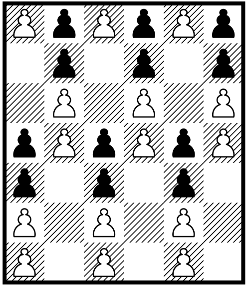


Wazir (W) and Ferz (F) attacks

https://exciting-hedgehog-6310.vibecode.garden/

Alternative boards

- Toroidal board:

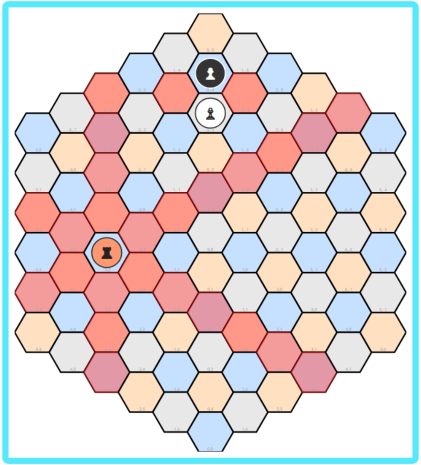


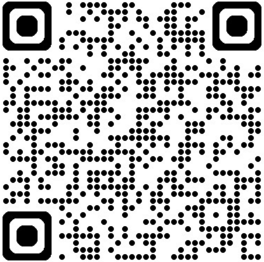
27 pawns on 7×6 torus board

https://round-bat-6533.vibecode.garden/

Alternative boards

- Hexagonal board:





https://fit-basilisk-8896.vibecode.garden/

Some references

- D. Chatham, How many mutually non-attacking, non-knight pieces can we place on an $m \times n$ chessboard?, submitted to *Recreational Mathematics Magazine*. https://www.overleaf.com/read/jrtgrzbnnqdh#0f4dd6
- The Chess Variant Pages. https://www.chessvariants.com
- M. J. Chlond, C. M. Toase, IP Modeling of Chessboard Placements and Related Puzzles. *INFORMS Transactions on Education* 2(2)(2002):1-11. https://doi.org/10.1287/ited.2.2.1
- M. Gardner, Some New Results on Nonattacking Chess Tasks, *Math Horizons*, **8**(3)(2001), 10–12. http://www.jstor.org/stable/25678300

More references

- G. Jelliss, Chess-Piece Arrangement Problems Part 4: Multiple Unguard Arrangements, *The Games and Puzzles Journal*, **39**(May-June 2005). https://www.mayhematics.com/a/arrangements4.htm
- J.J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton University Press, Princeton and Oxford. (2004) http://dx.doi.org/10.1515/9781400840922
- MiniZinc website. http://www.minizinc.org

Rectangular App	Toroidal App	Hexagonal App

Any questions?